Lecture 5: Wedderburn Decomposition and Group Algebras in Modular Settings

Goal: Understand the structure of group algebras over fields of characteristic 0 and characteristic p, study how semisimplicity fails when $p \mid |G|$, and learn about simple components, blocks, and how they relate to modular representations.

1. Group Algebras and Representations

Definition 5.1 (Group Algebra). Let G be a finite group and F a field. The *group algebra* $F[G]$ is the set of formal sums:

$$F[G] := \left\{ \sum_{g \in G} a_g g \;\middle|\; a_g \in F \right\},$$

with multiplication extended linearly from G.

Theorem 5.2. There is an equivalence of categories:

$$(\text{finite-dimensional representations of } G \text{ over } F) \quad \cong \quad (\text{finite-dimensional } F[G]\text{-modules}).$$

2. Semisimple Algebras and Maschke's Theorem

Theorem 5.3 (Maschke's Theorem). Let F be a field with $\text{char}(F) \nmid |G|$. Then the group algebra $F[G]$ is semisimple.

Definition 5.4 (Semisimple Algebra). An F-algebra A is *semisimple* if every A-module is semisimple (i.e., completely reducible).

3. Wedderburn-Artin Decomposition

Theorem 5.5 (Wedderburn-Artin). If A is a semisimple finite-dimensional F-algebra, then:

$$A \cong \bigoplus_{i=1}^{r} M_{n_i}(D_i),$$

where D_i are division algebras over F, and $M_{n_i}(D_i)$ denotes the algebra of $n_i \times n_i$ matrices over D_i.

Corollary 5.6. If $F = \mathbb{C}$, then every finite-dimensional semisimple $\mathbb{C}$-algebra is isomorphic to a direct sum of matrix algebras over $\mathbb{C}$:

$$\mathbb{C}[G] \cong \bigoplus_{i=1}^{r} M_{n_i}(\mathbb{C}),$$

where r equals the number of irreducible complex representations of G.

4. Failure of Semisimplicity in Characteristic p

Theorem 5.7. If $\mathrm{char}(F) = p \mid |G|$, then $F[G]$ is not semisimple.

Example 5.8. Let $G = \mathbb{Z}/p\mathbb{Z}$ and $F = \mathbb{F}_p$. Then:

$$F[G] \cong \frac{F[x]}{x^p - 1} = \frac{F[x]}{(x-1)^p}$$

since $x^p - 1 = (x-1)^p$ in $\mathbb{F}_p$. This ring is local and not semisimple.

5. Radical and Semisimple Quotients

Definition 5.9 (Jacobson Radical). Let A be a ring. The *Jacobson radical* $\mathrm{Rad}(A)$ is the intersection of all maximal left ideals. It is the largest ideal such that $A/\mathrm{Rad}(A)$ is semisimple.

Definition 5.10 (Modular Semisimple Quotient). For $F = \overline{\mathbb{F}}_p$, the semisimple quotient:

$$F[G]/\mathrm{Rad}(F[G])$$

determines the irreducible modular representations of G.

Theorem 5.11. There are only finitely many non-isomorphic simple modules over $F[G]$, and their number equals the number of p-regular conjugacy classes.

6. Examples

Example 5.12: S_3 over $\mathbb{F}_3$.

- $\mathrm{char}(F) = 3 \mid |S_3| = 6$
- $\mathbb{F}_3[S_3]$ is not semisimple
- Still has two simple modules φ_1, φ_2, as seen from the Brauer character table

7. Counterexamples

Counterexample 5.13. In $\mathbb{F}_p[G]$, not every module decomposes as a direct sum of simples. Indecomposable modules (e.g., uniserial modules) appear due to the failure of semisimplicity.

8. Summary

In this lecture we learned:

- The structure of group algebras via the Wedderburn-Artin decomposition
- Semisimplicity in characteristic 0 vs modular failure when $p \mid |G|$
- How the Jacobson radical captures the "non-semisimple" part
- Why modular representation theory is richer and more subtle than classical theory

Coming Up in Lecture 6: We will develop *block theory*, introducing the notions of *blocks*, *idempotents*, and *defect groups*, and how these relate to modular characters and decomposition matrices.